Perturbation Guidance Laws for Perfect Information Interceptors with Symmetrical Nonlinearities

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The nonlinearities in a dynamic system are assumed to be cubic and "small," i.e., all proportional to a single scalar small parameter ϵ. The optimal digital nonlinear feedback control law in the case of perfect state knowledge, but with dynamic noise input, is carried through the first power of ϵ. The control law involves cubic as well as linear terms in the state and depends on the prior covariance of the process noise. The theory is applied to a constant-speed coplanar model of a saturated interception and the numerical investigation shows a considerable advantage of the present nonlinear control law over the linear law ($\epsilon=0$) as applied to the same system.

I. Introduction

IN many cases, it is possible to represent the dynamics of a nonlinear stochastic interception system as a set of nonlinear state equations where the derivatives of the state variables are all odd functions of the states, the controls, and the dynamic noise terms. This class of cases is referred to in the present work as "symmetrical interceptors." In Ref. 1, a first-order perturbation method was applied to symmetrical stochastic controllers in order to yield, via dynamic programming, optimal feedback control laws for both the perfect information case and the fully stochastic situation. In the present paper, we use the perfect information part of that theory in order to derive optimal guidance laws for interception systems whose state vector is fully accessible but subjected to dynamic noise input. Although the perturbation method is mathemtically rigorous for small nonlinearities, our nonlinear law proves to be very useful for the rather substantial nonlinearities that appear in the presence of large state excursions from the origin.

Former works[2,3] successfully approached the continuous-time deterministic cases (no dynamic noise input) and had actually shown the derivation for elementary examples. The present work demonstrates the derivation of a digital control law for the general case, that is optimal to first order in the stochastic environment, i.e., when dynamic noise affects the plant. Some of the results of this article were briefly presented in Ref. 4.

II. First-Order Optimal State Feedback Control of Symmetrical Perfect Information Systems

Consider a set of nonlinear discrete state equations,

$$x(k+1)=f\{x(k),u(k),w(k),k\} \tag{1}$$

where x is the state vector, u the control vector, and w the dynamic noise input vector assumed to be purely random, with zero-odd-moments and known covariance.

The elements of the vector function f are all assumed odd in x, u, and w. Then, if we expand f into its Taylor series about the origin, retain terms up to first order only, and further exclude quadratic and cubic terms in w, we arrive at‡

$$x(k+1)=Fx+Gu+w+\epsilon[a_{i\ell mn}x_\ell x_m x_n+b_{i\ell mn}x_\ell x_m u_n + c_{i\ell mn}x_\ell u_m u_n+d_{i\ell mn}u_\ell u_m u_n+M_{i\ell mn}x_\ell x_m w_n] \tag{2}$$

In Eq. (2), the various quantities are i-indexed vectors and a hybrid vector-tensor notation is used where summation is understood over the repeating subscripts. ϵ is a small scalar and decoupling is assumed between w and u (although the theory can handle such coupling, as well as higher terms in w). All the variables at the right-hand side of Eq. (2) are given at the kth stage and the system's parameters F, G, a, b, c, d, and M can be nonstationary as well.

We have $E\{w_i w_j\}\triangleq Q_{ij}$, where Q is not necessarily stationary. The tensors a, b, c, d, and M are all symmetrical in similar variables.

A cost J is defined by

$$J\triangleq E\left\{\sum_{k=0}^{N-1}\left[\frac{1}{2}x^T(k)A(k)x(k)+\frac{1}{2}u^T(k)B(k)u(k)\right] +\frac{1}{2}x^T(N)S_f x(N)+\epsilon(\xi_f)_{\ell mnq}x_\ell(N)x_m(N)x_n(N)x_q(N)\right\} \tag{3}$$

where A, B, S_f and $(\xi_f)_{\ell mnq}$ are specified, thus the minimum cost to go from the kth stage satisfies the dynamic programming equation (DPE),[5]

$$J_k=\underset{u(k)}{\mathrm{Min}}\left\{E\left[\frac{1}{2}x^T(k)A(k)x(k) +\frac{1}{2}u^T(k)B(k)u(k)+J_{k+1}\right]\right\} \tag{4}$$

We make the four following statements:

1) A first-order solution to DPE (4) is

$$J_k=E\left[\frac{1}{2}x^T(k)S(k)x(k) +\epsilon\xi_{\ell mnq}(k)x_\ell(k)x_m(k)x_n(k)x_q(k)+D(k)\right] \tag{5}$$

where $D(k)$ is independent of the state and $\xi_{\ell mnq}$ a fourth-order symmetrical tensor.

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‡The need of a noise-matrix is avoided by rescaling of w such that:

$$\left.\frac{\partial f}{\partial w}\right|_{x,w=0}=1.$$

2) The cost parameters satisfy backward recursions given by

$$S(k) = A + C^T BC + (F-GC)^T S(k+1)(F-GC) + 12\epsilon (F-GC)^T [\xi_{\ell mnq}(k+1) Q_{nq}](F-GC) + 2\epsilon S_{ij}(k+1)(M_{\ell m})_{jn} Q_{in} \tag{6}$$

$$[\xi_{\alpha\beta\gamma\delta}(k)]_{p(\alpha\beta\gamma\delta)} = [S_{ij}(k+1)(F-GC)_{i\alpha}(a_{j\beta\gamma\delta} - b_{j\beta\gamma r})C_{r\delta} + c_{j\beta rs} C_{r\gamma} C_{s\delta} - d_{jrst} C_{r\beta} C_{s\gamma} C_{t\delta}) + \xi_{\ell mnq}(k+1)(F-GC)^4_{\substack{\ell\alpha\\ m\beta\\ n\gamma\\ q\delta}}]_{p(\alpha\beta\gamma\delta)} \tag{7}$$

with

$$C \triangleq [B + G^T S(k+1) G]^{-1} G^T S(k+1) F$$

and

$$S(N) = S_f; \quad \xi_{\ell mnq}(N) = (\xi_f)_{\ell mnq}$$

where $P(\cdot)$ is the summation over all the permutations of $(\cdot)$, and an additional recursion in D is obtainable (which is not needed for the present analysis).

3) The first-order optimal control is given by

$$u_q^* = -C_{qi} x_i - \epsilon [B + G^T S(k+1) G]^{-1}_{qp} \{ S_{ij}(k+1 \times [(F-GC)_{ir}(b_{jstp} - 2c_{jsmp} C_{mt} + 3d_{j\ell mp} C^2_{\substack{\ell t\\ ms}}) + G_{ip}(a_{jrst} - b_{jrsn} C_{nt} + c_{jrmn} C^2_{\substack{ms\\ nt}} - d_{j\ell mn} C^3_{\substack{\ell r\\ ms\\ nt}})] + 4\xi_{\ell mnz}(k+1) G_{\ell p}(F-GC)^3_{\substack{ms\\ nr\\ zt}} \} x^3_{rst} - 12\epsilon [B + G^T S(k+1) G]^{-1}_{qp} \xi_{\ell mns}(k+1) G_{\ell p} Q_{mn} \times (F-GC)_{si} x_i \triangleq -C_{qj} x_j - \epsilon(\phi_{qj} x_j + \psi_{qrst} x^3_{rst}) \tag{8}$$

4) The optimal control [Eq. (8)] is within the ϵ^2 vicinity of the true optimum and, hence, yields an actual cost differing from the true minimum by $\mathcal{O}(\epsilon^4)$.

The proof of statements 1-3 is by a direct substitution of Eq. (5) into Eq. (4), expressing $x(k+1)$ on the right-hand side in terms of x,u,w, using Eq. (2). Then, assuming mutual independence of w and x, w can be averaged using its known three lowest moments. (The fourth moment is needed for D-equation only, hence not required for the present analysis.) Then, the result is minimized with respect to u, truncated after $\mathcal{O}(\epsilon^1)$. This yields Eq. (8). In order to obtain Eqs. (6) and (7), the $\mathcal{O}(\epsilon^1)$ minimizing control of Eq. (8) must be put back into the averaged DPE. This step can be simplified by noting that for a minimizing control: $u^* \triangleq u^{(0)} + \epsilon u^{(1)}$, $J_k(u^{(0)})$ differs from $J_K(u^*)$ by $\mathcal{O}(\epsilon^2)$ only. Thus, $u^{(1)}$ does not affect the $\mathcal{O}(\epsilon^1)$ DPE and can be omitted for the purpose of obtaining Eqs. (6) and (7). These two recursions (together with a third one in D) can be extracted from the DPE by equating coefficients of second and fourth powers of state variables, since with the optimal control known the DPE becomes an identity in x, consisting of zero, second, and fourth powers. The obtainability of backward recursions in the cost parameters validates the solution [Eq. (5)] and proves statements 1-3. The end conditions of Eqs. (6) and (7) are readily obtained by substituting $k=N$ in the general expression of the cost to go. Statement 4 follows directly, since the cost is quadratic in the control.

We conclude this section by a few remarks:

1) Clearly, Eq. (8) is a feedback law, since by virtue of Eqs. (6) and (7) all the "gains" of Eq. (8) are precomputatble offline ahead of time.

2) Since the DPE is a scalar equation, S must be a symmetric matrix and ξ must be a symmetric fourth-order tensor.

3) Unlike linear controllers, our nonlinear control law depends, by virtue of Eqs. (6) and (8), on the prior covariance of the process noise.

4) For $\epsilon \to 0$, our optimal nonlinear solution converges to the solution that is obtained by the linear quadratic theory for discrete linear controllers.

III. Numerical Example: Constant Speed, Saturated Planar Interception with One Time Constant

In our example, we consider planar interception with perfect information, the velocities of both the target and the interceptor being constant in magnitude. The control variable is scalar and the system is clearly symmetrical about the line of sight (LOS). The goal in this example is to compare the performance of the nonlinear control law that is developed in Sec. II to that of a linear law obtained by the classical linear quadratic theory that neglects the nonlinearities in the plant. These originate in both the geometry of the pursuit and a saturation in the interceptor's response. We also assume one time constant in the interceptor's dynamics. This brings the order of the augmented system under consideration to three, with the input being the target acceleration, which is assumed to be a white random variable of zero-odd-moments. In Ref. 1, first-order statistical analysis algorithms were developed in order to evaluate the closed-loop performance of the optimally controlled nonlinear systems.

However, since the difference between the two control laws that are examined is of $\mathcal{O}(\epsilon^1)$, our first-order statistical analysis algorithm should not be able to show any difference in the cost. In fact, if we carefully assure that the analysis is strictly $\mathcal{O}(\epsilon^1)$, that is, no ϵ^2 or higher terms are implicitly carried along, the $\mathcal{O}(\epsilon^1)$ statistical analysis should give identical costs for the nonlinear plant driven by the linear and the nonlinear optimal laws. This makes our statistical analysis inappropriate for the particular comparison desired.

Since a second-order statistical analysis is beyond the scope of the present work, we choose to perform the comparison by a deterministic simulation with a constant target acceleration.

The example problem is illustrated in Fig. 1 where x-y are some arbitrary inertial coordinates, E the target, and P the interceptor (LOS = line-of-sight). Denoting the target acceleration by a_e and the interceptor acceleration by a_p, and with the assumptions that $a_p \perp v_p$ and $a_e \perp v_e$, the kinematic equations are

$$\dot{\lambda} = (1/R)[v_p \sin(\lambda - \gamma_p) - v_e \sin(\lambda - \gamma_e)] \tag{9a}$$

$$\dot{R} = v_e \cos(\lambda - \gamma_e) - v_p \cos(\lambda - \gamma_p) \tag{9b}$$

$$\dot{\gamma}_e = a_e / v_e \tag{9c}$$

$$\dot{\gamma}_p = a_p / v_p \tag{9d}$$

where the last two equations define the positive directions of the accelerations.

The interceptor dynamics is assumed to be given by

$$\dot{a}_p = -a_p/\tau_p + \mathrm{SAT}_{\ell}^{k_p}(u)/\tau_p \tag{10}$$

where τ_p is the interceptor's time constant, u the controlling command, and $\mathrm{SAT}_{\ell}^{k_p}$ the saturation function described in Fig. 2.

Now, for an ever-decreasing range situation, a transformation of the independent variable from time into range can be made. Equations (9) can also be written in coordinates attached to the LOS, using α and β. Transforming into range as independent variable by dividing each differential equation

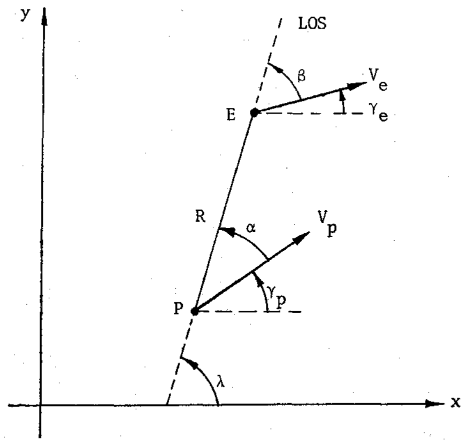

Fig. 1 Kinematics of the example problem.

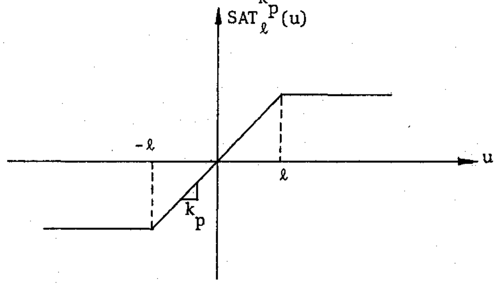

Fig. 2 Saturation function of the interceptor.

by $\dot{R}$ and writing the equations of motion in the LOS system, we finally obtain

$$\alpha' = \frac{v_p s\alpha - v_e s\beta}{R(v_e c\beta - v_p c\alpha)} - \frac{a_p}{v_p(v_e c\beta - v_p c\alpha)} \tag{11a}$$

$$\beta' = \frac{v_p s\alpha - v_e s\beta}{R(v_e c\beta - v_p c\alpha)} - \frac{a_e}{v_e(v_e c\beta - v_p c\alpha)} \tag{11b}$$

$$a_p' = \frac{-a_p}{\tau_p(v_e c\beta - v_p c\alpha)} + \frac{\mathrm{SAT}_\ell^{k_p}(u)}{\tau_p(v_e c\beta - v_p c\alpha)} \tag{11c}$$

where several shorthand notations have been used: $s\alpha \triangleq \sin\alpha$, $c\alpha \triangleq \cos\alpha$, etc., and $\alpha' \triangleq \mathrm{d}\alpha/\mathrm{d}R$, etc.

Representing the saturation by a cubic polynomial with the same slope k_p at the origin and the same extreme values $\pm k_p \ell$,

$$\mathrm{SAT}_\ell^{k_p}(u) \cong k_p u + \mathrm{d}u^3; \quad \mathrm{d} = -\frac{4}{27}\frac{k_p}{\ell^2} \tag{12}$$

In order to use the standard state-space notation, we denote $\alpha \triangleq x_1$, $\beta \triangleq x_2$, and $a_p \triangleq x_3$ and use Eq. (12) to expand Eq. (11) into its four-term Taylor series about the origin of the state-space. The resulting system is then discretized using $\Delta R = -1m$ and the chosen parameters

$$v_p = 500\ \mathrm{m/s}, \quad v_e = 200\ \mathrm{m/s}, \quad \tau_p = 0.02\ \mathrm{s}$$

$$\tau_e = 0.2\ \mathrm{s}, \quad \sigma_e^2 = (50\ \mathrm{m/s^2})^2$$

$$k_p = 1, \quad \ell = 1000$$

where τ_e is the target corelation time and σ_e^2 is the square of expected target acceleration. This parameter represents the difficulty of the scenarios for which the system is designed.

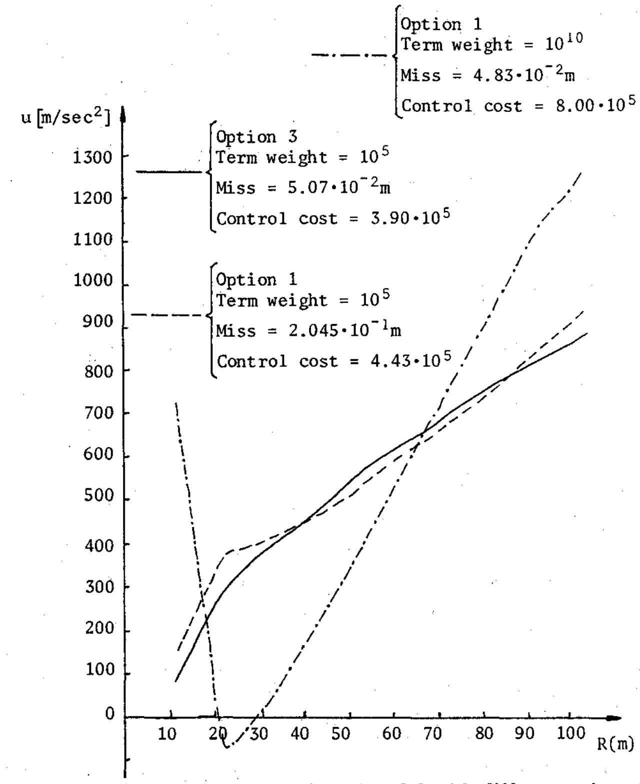

Fig. 3 Control histories of options 1 and 3 with different gains at $\sigma_e^2 = (50\ \mathrm{m/s^2})^2$.

With some algebra, the result is

$$F = \begin{bmatrix} \left(1+\frac{5}{3R}\right) & \left(-\frac{2}{3R}\right) & \left(-\frac{2}{3}\cdot 10^{-5}\right) \\ \left(\frac{5}{3R}\right) & \left(1-\frac{2}{3R}\right) & 0 \\ 0 & 0 & \left(1-\frac{1}{6}\right) \end{bmatrix}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{6} \end{bmatrix}; \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\sigma_e^2}{6\cdot 10^7} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a_{1111} = a_{2111} = 10/9R\epsilon$$

$$a_{1112} = a_{1121} = a_{1211} = -5/27R\epsilon$$

$$a_{2112} = a_{2121} = a_{2211} = -5/27R\epsilon$$

$$a_{1221} = a_{1212} = a_{1122} = -5/27R\epsilon$$

$$a_{1113} = a_{1131} = a_{1311} = -10^{-4}/54\epsilon$$

$$a_{1222} = a_{2222} = 1/3R\epsilon$$

$$a_{1223} = a_{1232} = a_{1322} = 10^{-5}/13.5\epsilon$$

$$a_{3113} = a_{3131} = a_{3311} = -5/108\epsilon$$

$$a_{3223} = a_{3232} = a_{3322} = 1/54\epsilon, \qquad \epsilon b_{311} = 5/36$$

$$\epsilon b_{322} = -1/18, \qquad \epsilon d_3 = -(1.4815/6)\cdot 10^{-7}$$

$$\epsilon M_{2112} = 5/6, \qquad \epsilon M_{2222} = -1/3$$

All the rest of the terms that correspond to those of Eq. (2) are zero (e.g., there are no c terms in this example). In the augmented target-interceptor system that results, the input is the target acceleration a_e, modeled here as a zero-mean white random variable. In such a stochastic environment, it would not be feasible to pose $R_f = 0$ as the end of the process. Instead, we chose some small but positive value for R_f that can be reached by the system using finite effort and with $\dot{R} < 0$ always. Then, approximate minimization of the miss distance is achieved by an appropriate choice of a terminal cost, combined with an open-loop control for the post-R_f phase. To simplify the example, we assume zero relative target-interceptor acceleration during the post-R_f phase and approximate the constant closing velocity that results by $(v_e - v_p)$.

Then if the rate of rotation of the LOS is neglected at $R = R_f$, the absolute value of the miss distance is given approximately by

$$R_{\text{miss}} = \left(\frac{R_f}{v_p - v_e}\right) \cdot |(v_e s\beta - v_p s\alpha)|_f \tag{13}$$

Hence, our terminal cost is chosen to be:

$$\text{Term cost} = \text{term weight} \cdot \tfrac{1}{2}(v_e s\beta - v_p s\alpha)_f^2 \tag{14}$$

Expanding both $s\alpha$ and $s\beta$ into their power series of two nonzero terms and with our standard state-space notation, we obtain $\mathcal{O}(\epsilon^1)$,

$$\text{Term cost} = \text{TW} \cdot \tfrac{1}{2}(x_1, x_2) \begin{bmatrix} v_p^2 & -v_e v_p \\ -v_e v_p & v_e^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ 1/6[-v_p^2 x_1^4 - v_e^2 x_2^4 + v_e v_p (x_1^3 x_2 + x_2^3 x_1)]\}_f$$

where TW = term weight. This implies the following end conditions on S and ξ, cf Eqs. (6) and (7), respectively:

$$S_{11}(R_f) = \text{TW} \cdot v_p^2 = 2.5 \cdot 10^5 \cdot \text{TW}$$

$$S_{22}(R_f) = \text{TW} \cdot v_e^2 = 4 \cdot 10^4 \cdot \text{TW}$$

$$S_{12}(R_f) = S_{21}(R_f) = \text{TW} \cdot v_e v_p = -10^5 \cdot \text{TW}$$

$$\epsilon \xi_{1111}(R_f) = -\text{TW} \cdot \frac{v_p^2}{6} = -\frac{2.5}{6} \cdot 10^5 \cdot \text{TW}$$

$$\epsilon \xi_{2222}(R_f) = -\text{TW} \cdot \frac{v_e^2}{6} = -\frac{2}{3} \cdot 10^4 \cdot \text{TW}$$

$$\xi_{1222}(R_f) = \xi_{2122}(R_f) = \xi_{2212}(R_f) = \xi_{2221}(R_f)$$

$$= \xi_{2111}(R_f) = \xi_{1211}(R_f) = \xi_{1121}(R_f) = \xi_{1112}(R_f)$$

$$= \frac{v_e v_p}{\epsilon 24} \cdot \text{TW} = \frac{10^5}{24\epsilon} \cdot \text{TW}$$

A computer program has been written in order to solve the example problem. The program solves Eqs. (6-8) and uses a strict ϵ^1 statistical analysis as a partial check. Finally, the program performs deterministic simulation of three versions (options):

1) The nonlinear system controlled by the linear law with the gain given by the first part of Eq. (8)—a "linearized system."

2) The nonlinear system controlled by a linear law that is given by the sum of the linear terms of Eq. (8)—an "intermediate law."

Table 1 Simulation cost and miss for options 1 and 3 at $\sigma_e^2 = (50\ \text{m/s}^2)^2$

Option	Simulation control cost	Simulation miss, m
1	$4.43 \cdot 10^5$	$2.05 \cdot 10^{-1}$
3	$3.90 \cdot 10^5$	$5.07 \cdot 10^{-2}$

Table 2 Simulation cost and miss for options 1-3 at $\sigma_e^2 = (350\ \text{m/s}^2)^2$

Option	Simulation control cost	Simulation miss, m
1	$4.43 \cdot 10^5$	$2.05 \cdot 10^{-1}$
2	$4.95 \cdot 10^5$	$1.09 \cdot 10^{-1}$
3	$4.40 \cdot 10^5$	$2.26 \cdot 10^{-3}$

3) The nonlinear system controlled by the full nonlinear law as defined by Eq. (8).

The simulation consists of discretizing Eq. (11) with $\Delta R = -1$ m. Using constant target acceleration, the simulation computes the actual cost from the actual squares and fourth powers of the states, and the actual miss by Eq. (13).

In order to illustrate the difference between the various options, the progam has been run from $R_0 = 102$ m to $R_f = 10$ m with a zero weight on the running state, i.e., $A = 0$. Also,

Control weight $B = 1.0$.
Terminal Weight $= 10^5$
a_e (constant for simulation) $= 50\ \text{m/s}^2$
Initial $\alpha^2 = 0.1$
Initial $\beta^2 = 0.1$
Initial interceptor square acceleration $a_p^2 = 0.01\ (\text{m/s}^2)^2$.

The results are given in Table 1 for options 1 and 3 when option 2 is practically indistinguishable from option 1. Table 1 shows that with about 10% less control effort, the nonlinear law manages to decrease the miss to about one-fourth of that obtained with the linear law. The control histories are illustrated in Fig. 3.

To further improve the performance for more "difficult" expected scenarios, we increase σ_e to $350\ \text{m/s}^2$. This yields the results shown in Table 2, for the same simulation conditions as above, which shows two orders of magnitude of improvement in the miss between options 1 and 3, per same control effort. More simulations for this particular design show that, with options 1 and 2 and any control effort chosen (use any desired terminal weight), it is still impossible to obtain a miss that is smaller than 10 times of that of option 3.

To complete the comparison with the "mild" situation $[\sigma_e^2 = (50)^2]$, the loop gain of the linearized option is raised to the point where the miss is comparable to that of the nonlinear law with the lower gain (raising the gain is done by increasing the terminal weight). With terminal weight $= 10^{10}$ in the linearized system, a miss of $4.83 \cdot 10^{-2}$ m is obtained with a simulation control cost of $8.00 \cdot 10^5\ (\text{m/s})^2$ m. The corresponding control history is illustrated in Fig. 3, with the associated control cost being about twice as large as that with the lower gain in both the linearized and the nonlinear systems.

With the last, much higher gain in the linearized system, the loop is actually nearing instability, which is indicated by the sign reversals of the control near the end. This property can be disadvantageous in the presence of parameter uncertainties. For the particular scenarios examined, we conclude that the nonlinear law seems to be much more efficient than the linear law used in the nonlinear engagement. We can judge the actual complexity involved in applying the nonlinear law by looking at the cubic feedback terms $\psi_{irst} x_{rst}^3$. Due to symmetry, there are 10 different coefficients (instead of 27) to deal with. At $R = 50$ m and terminal weight $= 10^5$, we have§

§The dimensions of these terms are that of the control.

$$|\psi_{1111}x_1^3| = 1.66\cdot 10^3$$

$$|\psi_{222}x_2^3| = 5.85\cdot 10^2$$

$$|\psi_{333}x_3^3| = 1.261$$

$$|(\psi_{112}+\psi_{121}+\psi_{211})x_1^2x_2| = 3.69\cdot 10^3$$

$$|(\psi_{113}+\psi_{131}+\psi_{311})x_1^2x_3| = 4.40\cdot 10^2$$

$$|(\psi_{223}+\psi_{232}+\psi_{322})x_2^2x_3| = 2.10\cdot 10^2$$

$$|(\psi_{221}+\psi_{212}+\psi_{122})x_2^2x_1| = 2.22\cdot 10^3$$

$$|(\psi_{331}+\psi_{313}+\psi_{133})x_3^2x_1| = 4.04\cdot 10^1$$

$$|(\psi_{332}+\psi_{323}+\psi_{233})x_3^2x_2| = 2.96\cdot 10^1$$

$$|(\psi_{123}+\psi_{132}+\psi_{213}+\psi_{231}+\psi_{312}+\psi_{321})x_1x_2x_3| = 6.57\cdot 10^2$$

This suggests¶ that one might try to omit the terms with x_3^3 and x_3^2 in order to obtain about the same performance with 7 terms in the nonlinear feedback, instead of 10.

In order to appreciate the additional computation effort, as compared to the linear law, we recall that in the latter we have to store three gain histories which are to be then multiplied by the three state variables in real time. This would typically require 2000 bytes of read only memory plus 2000 operations per stage. The additional effort that is needed by our nonlinear law is to store seven gain histories, to be further multiplied by seven cubic products of state variables in real time. This would require about 20,000 bytes of read only memory plus 60,000 operations per stage, which may be too close to the maximum power of the present class of processors that are used for these purposes. Thus, for higher-order systems, it would probably be worth trying, either by statistical analysis or extensive simulations, to check for the possibility of omitting more cubic feedback terms without cutting too much out of the improvement.

¶The ratios between the various ψx^3 terms are kept at the same orders of magnitude for other ranges and design conditions. However, it is recommended this be rechecked for every new design.

IV. Conclusions

We have applied a perturbation method to a class of nonlinear stochastic processes. This yields a first-order digital control in the form of a feedback law. The solution consists of cubic, as well as linear, terms in the state variables and converges to the result of the optimal linear quadratic theory when the system's nonlinearities approach zero. The present nonlinear control law depends on the prior covariance of the process noise. Thus, it enables the designer to account for the expected difficulty of the task in the solution of homogeneous stochastic problems, such as guidance toward a target whose maneuvering is unpredictable. A numerical example of a planar interception with substantial nonlinearities has been presented, in which the nonlinear control law is compared to the linear optimal controller applied to the same plant. Our nonlinear law shows a considerably better performance in simulations of deterministic scenarios. This improvement is further increased significantly by taking the target maneuverability into account in the design of the nonlinear control.

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